CIMAT

Values for teams games

L. Hernández-Lamoneda and F. Sánchez-Sánchez

SING 2 - Spain Italy Netherlands Conference on Game Theory

CIMAT

1. Introduction

Let G be the set of TU cooperative games with space of players $N = \{1, \ldots, n\}$.

For every $s : 1, \ldots, n$, let

$$G_s = \{v \in G \mid v(S) = 0 \text{ if } |S| \neq s\};$$

we think of the elements in G_s as “team games” with s players.

Problem. We want to divide an amount c in a “fair” way, among the players in a team game. One wants that each player’s share depends on his performance, although we take into account only the coalitions’ performance.

By a solution on $(v, c) \in G_s \oplus \mathbb{R}$ we mean a function

$$\varphi : G_s \otimes \mathbb{R} \to \mathbb{R}^n.$$

Example 1. We could see the bankruptcy game $[c, x]$ as an element (x, c) of $G_1 \otimes \mathbb{R}$, where $v(\{i\}) = x_i$.

Example 2. A Market with m sellers and n buyers.
M set of sellers.
N set of buyers.
1 product.
Assumption: $m \leq n$

$$v(\{i, j\}) = \begin{cases} 1 & i \text{ seller and } j \text{ buyer} \\ 0 & \text{otherwise} \end{cases}$$

CIMAT

Example 3. Assignment Game (Shapley and Shubik). Market in private homes.

m sellers.

n buyers.

c_i the value that seller gives to his house i.

h_{ij} the value that buyer j gives to the house i.

For a seller i and buyer j,

$$v(\{i,j\}) = \max\{0, h_{ij} - c_i\} = a_{ij}.$$

CIMAT

Example 4. When a traveler purchases a Europass he can choose to visit a subset of s countries (where $s = 3, 4$ or 5) from a set of N European countries, to travel for a certain fix amount of days d. We could think of a game (v, c) where $v \in G_s$ is such that $v(S)$ = the number of travelers (in a particular summer, say) that chose the set S of countries for his pass, and where c is the total amount collected, at the end of the period, from Europasses purchased.

Then a solution

$$\varphi : G_s \oplus \mathbb{R} \to \mathbb{R}^N$$

will assign to (v, c) the vector $\varphi(v, c)$ where $\varphi_j(v, c)$ is the amount that corresponds to country j.

CIMAT

Example 5. (Poker)

$N = \{1, \ldots, 52\}$ set of cards.

S vs T with $|S| = |T| = 5$.

$v(S) =$ Expected value.

$\varphi_j(v, 0)$ value of card j.

CIMAT

2. General results

Proposition 1. *The space of linear, symmetric solutions*

$$\varphi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$$

is 3-dimensional. Moreover, their general expression is given by

$$\varphi_i(v,c) = \alpha c + \beta \sum_{S \ni i} v(S) - \gamma \sum_{S \not\ni i} v(S) \tag{2.1}$$

for arbitrary $\alpha, \beta, \gamma \in \mathbb{R}$ *and* $i \in N$.

CIMAT

Definition 1. *(Efficiency Axiom) The solution* $\varphi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ *is said to be efficient if*

$$\varphi(v, c) \cdot \; 1_n = c.$$

Teorema 1. *The space of linear, symmetric solutions* $\varphi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ *that are also efficient is 1-dimensional. Their general expression is given by*

$$\varphi_i(v, c) = \frac{c}{n} + \lambda \left[\sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n - s} \right]$$

for arbitrary $\lambda \in \mathbb{R}$.

CIMAT

For every $x \in \mathbb{R}^n$, let

$$x^s(S) = \begin{cases} x(S) & \text{if } |S| = s \\ 0 & \text{otherwise} \end{cases}$$

where $x(S) = \sum_{i \in S} x_i$.

Definition 2. *(Naturalness Axiom) The solution $\varphi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ is said to be natural if*

$$\varphi(x^s, x \cdot \; 1_n) = x.$$

for every $x \in \mathbb{R}^n$.

Teorema 2. *There exists a unique solution $\psi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ which is linear, symmetric, efficient and natural. It is given by*

$$\psi_i(v,c) = \frac{c}{n} + \frac{n-1}{\binom{n}{s}} \left[\sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n-s} \right].$$

Remark *The formula for the unique linear, symmetric, efficient and natural solution, ψ, on $G_s \oplus \mathbb{R}$ takes the form:*

$$\psi(v,c) = \frac{c}{n} 1_n + (n-1)Sh(v).$$

Let us suppose that the amount to be share is equal to the sum of all the worth's coalitions, then it results reasonably that every player gets the proportional part of what he help to generate. So,

CIMAT

Axiom *(Collectability) We say that φ satisfies the collectable axiom if*

$$\varphi_i(v, \sum_S v(S)) = \sum_{S \ni i} \frac{v(S)}{s}.$$

Teorema 3. *There exist a unique linear, symmetric, efficient and collectable solution, φ, on $G_s \oplus \mathbb{R}$. Furthemore it takes the form:*

$$\varphi_i(v, c) = \frac{c}{n} + \frac{n-s}{n}\left[\sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n-s}\right].$$

CIMAT

3. Bankruptcy problem

- There is a 3-dimensional space of solutions $\varphi : G_1 \oplus \mathbb{R} \to \mathbb{R}^n$ for the bankruptcy problem. The general expression is given by

$$\varphi_i(x, c) = \alpha c + \beta x_i - \gamma \sum_{k \neq i} x_k$$

- There is a 1-dimensional space of linear symmetric and efficient solutions given by

$$\varphi_i(x, c) = \frac{c}{n} + \widetilde{\lambda}\,(x_i - \overline{x})$$

where $\overline{x}$ is the average of the coordinates of x.

- There exists a unique bankruptcy solution which is linear, symmetric, efficient and natural. It is given by

$$\varphi_i(x, c) = \frac{c}{n} + x_i - \overline{x}$$

CIMAT

4. Assignment Game

The solution of theorem 2 takes the following form,

$$\varphi_i(A) = \sum_k \frac{a_{ki}}{2} - \sum_k \sum_{j \neq i} \frac{a_{kj}}{2n-2}$$

5. A Market with 1 sellers and 2 buyers

The solution is: $(1, 0, 0)$ the only imputation in the core.

6. Poker

j	$\varphi_j(v,0)$
A	0.236088
K	0.128311
Q	0.066628
J	0.029374
10	0.005330
9	-0.012679
8	-0.027257
7	-0.040026
6	-0.051830
5	-0.063794
4	-0.076991
3	-0.090044
2	-0.103109

CIMAT

References

Hernández-Lamoneda L., Juárez-García R. and Sánchez-Sánchez F. "Dissection of cooperative solutions in game theory using representation techniques", preprint.

Kultti K. and Salonen H. 2005. "Minimum Norm Solutions for Cooperative Games", preprint.

Saari D.G. and Sieberg K.K. (2001). "Some Surprising Properties of Power Indices", *Games and Economic Behavior*, vol. 36, 2, pp241-263.

Shapley L.S. and Shubik M. (1972) "The Assignment Game I: The Core", International Journal of Game Theory, 111–130.